# Invariants

## Introduction

Our brains have an amazing capacity to recognise objects as being the same even after they have been flipped, rotated, moved or otherwise distorted. Similarly, in mathematics a quantity or relationship can remain unchanged after a mathematical operation or transformation is applied. This idea is known as invariance and is rooted in many areas of mathematics including algebra, geometry and trigonometry, in addition to computer science. Angles, for example, are invariant under the operations of rotation and reflection. In other words, the angle will still be the same when a rotation or reflection is applied. The ability to recognise such invariants and to decompose problems in mathematics are important skills which will be explored in this workshop.

## Aim of Workshop

The aim of this workshop is to introduce students to the concept of invariants through a series of puzzles. Students will also be encouraged to decompose the problems as a series of variables and to identify the invariant.

## Learning Outcomes

By the end of this workshop, students will be able to:

- Describe, in their own words, what is meant by invariance
- Solve problems involving invariants

## Materials and Resources

Per Group: Scissors, Sellotape, nets of shapes, 7 cups, a bar of chocolate (different numbers of squares are optional)

## KEY WORDS

**Parity**
The state of a number being even or odd

**Invariant**
An expression or quantity which remains unchanged after a mathematical operation or transformation is applied.

## Invariants: Workshop Outline

| Suggested Time (Total mins) | Activity | Description |
|---|---|---|
| 5 mins<br>(00:05) | Introduction to Invariance | · Introduce the concept of invariance and provide examples (see **Workshop Introduction**)<br>· Compare an invariant to a variable |
| 15 mins<br>(00:20) | Activity 1<br>Nets and Cubes | · Hand out the nets, Sellotape and scissors to each group (see **Appendix – Note 1**)<br>· **Activity 1:** Students construct the shapes using Sellotape and try to identify the least number of cuts to turn the shapes back to nets again (see **Appendix – Note 2**)<br>· Encourage students to compare results with each other. Is there a relationship between the shapes and the number of cuts? |
| 15 mins<br>(00:35) | Activity 2<br>Cups | · Explain the rules of the cup game (see **Appendix – Note 3**)<br>· **Activity 2:** In pairs or small groups, students try to find the initial states where it is possible to turn all the cups the right way up (see **Appendix – Note 4**) |
| 15 mins<br>(00:50) | Activity 3<br>Handshake Game | · Explain the rules of the handshake game (see **Appendix – Note 5**)<br>· **Activity 3:** Students shake hands with each other for 30 seconds and count how many hands they shook in total. Why is there an even number of people with odd handshakes?<br>· Demonstrate the results to the class (see **Appendix – Note 6**) |
| 10 mins<br>(01:00) | Activity 4<br>All the Chocolate | · Explain the rules of the game (see **Appendix – Note 7**)<br>· **Activity 4:** In groups, students try to determine the number of cuts it takes to break a chocolate bar into individual pieces (see **Appendix – Note 8**) |

## Invariants: Workshop Appendix

### Note 1: Nets of Shapes

Students create a variety of three-dimensional shapes from the nets and try to determine the least number of cuts that are needed to turn the shapes back to nets again. You may wish to encourage the students to consider the different characteristics of the shape (i.e. faces, edges, and vertices) and see if they can identify any relationships. See **Activity 1** for net templates.

### Note 2: Solution for Nets of Shapes

There is a relationship between the number of vertices (v) and the cuts (c) required to reduce the shape back to its net form. As the number of vertices increases so too does the number of cuts. This can be represented as $v \to v + 1, c \to c + 1$

This means that the difference between the vertices and cuts required is invariant. The minimum number of vertices to form a three-dimensional shape is four. For example, it takes three cuts to reduce a triangular based pyramid to its net. The difference between vertices can, therefore, be represented by $v - 1 = c$

### Note 3: Seven Cups

Seven cups are placed in a line on the table with some cups positioned upside down and some the right way up. The students must move all the cups the right way up. However, the cups may not be turned individually; you are only allowed to turn any two cups simultaneously. Encourage students to consider from which initial states of the cups is it possible to turn all the cups the right way up.

### Note 4: Solution for Seven Cups

This activity concerns the number of upward cups $u$.

There are three possible moves for the cups:

- Flip two downward facing cups up. This has the effect of $u \to u + 2$
- Flip two upward facing cups down: $u \to u - 2$
- Flip one cup from up to down, and one from down to up: $u \to u$

No matter what move we make, the parity of the upward cups is conserved. So, if $u$ is initially even, there is no move we can do to make this odd. Consequently, $u$ will never equal seven. If $u$ is initially odd however, we can continue flipping downward cups until $u = 7$.

### Note 5: Handshake Game

Students move around the classroom and shake hands for 30 seconds whilst ensuring to keep track of the number of handshakes. After 30 seconds, count the number of people with an odd number of handshakes. The number of people with odd handshakes should be even. Repeat the handshaking process again. The number of people with odd handshakes should still be even. Encourage students to consider why there is an even number of people with odd handshakes.

## Note 6: Solution for Handshake Game

Ask the students with an even number of handshakes (e) to stand at one end of the table and all the students with an odd number of handshakes (o) to stand at the other end. Demonstrate the possible moves:

- Have two evens shake hands and instruct them to go to the other side of the table as their new number of handshakes is now odd:

  $e \rightarrow e - 2,\ o \rightarrow o + 2$

- Have the odds shake hands an instruct them to go to the other side:

  $e \rightarrow e + 2, o \rightarrow o - 2$

- Have an odd and an even shake hands and instruct them to swap sides:

  $e \rightarrow e,\ o \rightarrow o$

Highlight at the beginning that everyone starts with zero handshakes and thus we begin with an even number (0) of people with an odd number of handshakes. Since the number of handshakes can only change by two, the parity is conserved.

## Note 7: All the Chocolate

Pass around a chocolate bar with divisions, giving everyone a chance to make a cut along the lines of the divisions. Keep track of the number of cuts. How many cuts does it take to break a chocolate bar into individual pieces? Prompt the students to consider the possible moves.

## Note 8: Solution for All the Chocolate

You begin with a single piece of chocolate. As you make a cut, you increase the number of pieces by one. This can be expressed as $n \rightarrow n + 1, c \rightarrow c + 1$

This means that the difference between the number of pieces and cuts required is invariant. We initially have one piece of chocolate and zero cuts, so the difference is one $n - 1 = c$. If you start with 16 pieces of chocolate, then it will take 15 cuts to break the bar into individual pieces.

## Invariants: Activity 1